ON THE STATIONARY STATES OF CONTROLLED SYSTEMS

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A control system with nonlinear feedback is considered. The origin is a stable equilibrium state of this system. Necessary and sufficient conditions are obtained under wich the system has more than one stationary state. The question of the stability of the "superfluous" equilibrium states is investigated. Certain elementary properties of the origin's region of attraction are examined.

1. Statement of the problem. We consider a control system described by the following matrix differential equation with constant coefficients:

$$dx / dt = Ax + bu \tag{1.1}$$

Here $x = \| x_i \|$, $A = \| a_{ij} \|$, $b = \| b_i \|$ are matrices of orders $(n \times 1)$, $(n \times n)$, $(n \times 1)$, respectively, u is the control function having the form

$$u = \varphi(\sigma) \qquad (\sigma = cx) \tag{1.2}$$

Here $c = \| c_i \|$ is a constant matrix of order $(1 \times n)$ and $\varphi(\sigma)$ is a continuous function satisfying the following constraints:

$$\varphi(0) = 0, \quad \varphi'(0) = k \quad (0 < k < \infty) \tag{1.3}$$

$$0 < \varphi(\sigma) / \sigma \leqslant k \quad \text{when } \sigma \neq 0 \tag{1.4}$$

$$\varphi(\sigma) / \sigma \to 0 \quad \text{as} \quad |\sigma| \to \infty \tag{1.5}$$

We shall assume that the usual existence and uniqueness conditions of the solutions under any initial conditions are fulfilled for the system (1.1), (1.2).

As a consequence of condition (1.3) the origin will be an equilibrium state of system (1.1), (1.2). Let us assume that this state is an asymptotically stable equilibrium position of system (1.1) when $u = k\sigma$, i.e. of the system

$$dx / dt = (A + kbc)x \tag{1.6}$$

In other words, let

$$\operatorname{Re} \lambda_i < 0 \qquad (i = 1, 2, \ldots, n) \tag{1.7}$$

where the λ_i $(i = 1, 2, \ldots, n)$ are the roots of the characteristic equation

$$\det (A + kbc - \lambda E) = 0 \tag{1.8}$$

As is easily seen [1], under condition (1.7) the state $x = 0$ is an asymptotically stable stationary state of system (1.1), (1.2).

We pose the problem: determine the conditions under which system (1.1), (1.2) has stationary states not just at the origin, i.e. has more than one stationary state. In other words, the problem consists in determining the conditions under which "superfluous" (unnecessary) equilibrium positions arise in system (1.1), (1.2). In the case when more than one stationary state are found in the system, we consider the question of the stability of these states.

The question on the number of equilibrium states and on their stability will be examined below also for the case when the function $\varphi(\sigma)$ is discontinuous at $\sigma = 0$ $(k = \infty)$.

In this case the constraints imposed on the function $\varphi(\sigma)$ will be somewhat modified.

2. Stationary states in the case $k < \infty$. It is obvious that those and only those states which satisfy the conditions

$$Ax = -b\varphi(cx) \tag{2.1}$$

are the equilibrium positions of system (1.1), (1.2).

Let us consider the case when the matrix A has zero eigenvalues ($\det(A) = 0$). Here the rank r of matrix A is less than n,

$$r(A) < n \tag{2.2}$$

We prove that the following lemma holds.

Lemma. Under the condition $\det(A) = 0$ the system (1.1), (1.2) has a unique equilibrium state at the origin.

Let

$$r\|b, Ab, \ldots, A^{n-1}b\| = \rho \leqslant n \tag{2.3}$$

When $\rho = n$, system (1.1) will be completely controllable in Kalman's sense [2].

As follows, for example, from [3, 4], system (1.1), (1.2) can be reduced by means of the nonsingular transformation $x = Qz$ to the form

$$dz_1/dt = A_1z_1 + A_3z_2 + b_1\varphi(c_1z_1 + c_2z_2), \quad dz_2/dt = A_2z_2 \tag{2.4}$$

Here $z_1, z_2, A_1, A_2, A_3, b_1, c_1, c_2$ are matrices of orders $(\rho \times 1)$, $[(n-\rho) \times 1]$, $(\rho \times \rho)$, $[(n-\rho) \times (n-\rho)]$, $[\rho \times (n-\rho)]$, $(\rho \times 1)$, $(1 \times \rho)$, $[1 \times (n-\rho)]$, respectively. Moreover,

$$r(V) = r\|b_1, A_1b_1, \ldots, A_1^{\rho-1}b_1\| = \rho \tag{2.5}$$

From the condition that system (2.4) is asymptotically stable it follows that $r(A_2) = n - \rho$, and, hence, from condition (2.2) we obtain

$$r(A_1) < \rho \tag{2.6}$$

From the equality $r(A_2) = n - \rho$ it follows that Eq. (2.1) in the variables z_1, z_2 acquires the form

$$A_1z_1 = -b_1\varphi(c_1z_1), \quad z_2 = 0 \tag{2.7}$$

Thus, the investigation of Eq. (2.1) is reduced to the investigation of Eq. (2.7) of order ρ, for which equality (2.5) holds.

The columns of matrix V are linear combinations of the columns of the matrix $\|A_1, b_1\|$ and, therefore, $r(V) \leqslant r\|A_1, b_1\|$ (see [5]). Since $r\|A_1, b_1\| \leqslant \rho$, we conclude from relation (2.5) that

$$r\|A_1, b_1\| = \rho \tag{2.8}$$

As is seen from (2.6) and (2.8), note that $r(A_1) = \rho - 1$.

Relations (2.6) and (2.8) form the foundation for the assertion that equality (2.7) can hold only when $\varphi(c_1z_1) = 0$. From condition (1.3) and inequality (1.4) it follows that $\varphi(\sigma) = 0$ only when $\sigma = 0$. Therefore, Eq. (2.7) can have only such solutions z_1 for which $c_1z_1 = 0$. Consequently, these solutions z_1 satisfy the equations

$$A_1z_1 = 0, \qquad c_1z_1 = 0 \tag{2.9}$$

If system (2.9) has even one nontrivial solution, then it has nontrivial solutions arbitrarily close to the origin. This signifies that system (2.4) has stationary states arbitrarily close to the origin, but this contradicts the condition that the state $x = 0$ is asymptotically stable. Consequently, Eq. (2.9) has only a trivial solution. Hence it follows that Eq. (2.7), and, therefore, also Eq. (2.1), has only a trivial solution.

Let us now assume that

$$r(A) = n \tag{2.10}$$

Then all the stationary states of system (1.1), (1.2) satisfy the matrix relation

$$x = -A^{-1}b\varphi(cx) \tag{2.11}$$

i.e. are located on a straight line passing through the origin. If a state x satisfies condition (2.11), then the quantity $\sigma = cx$ satisfies, obviously, the scalar equation

$$\sigma = -cA^{-1}b\varphi(\sigma) \tag{2.12}$$

The converse is also easily shown: if a certain number σ is a solution of Eq. (2.12), then the vector

$$x = -A^{-1}b\varphi(\sigma) \tag{2.13}$$

is a solution of Eq. (2.11).

Thus, the matrix equation (2.11) has as many solutions as does the scalar equation (2.12). Therefore, in order to solve the problem we have posed it suffices to investigate Eq. (2.12). The number of roots of Eq. (2.12) depends on what the quantity $cA^{-1}b$ itself represents. Our judgement on this quantity, needed for solving the problem being considered, can be made by using the stability condition (1.7) for the linear system (1.6).

As follows from the determinantal relation derived, for example, in [3] (p.132), the characteristic equation (1.8) can be represented in the form

$$\det(A - \lambda E)\,[1 + kc(A - \lambda E)^{-1}b] = 0 \tag{2.14}$$

For the fulfillment of inequalities (1.7) it is necessary that the sign of the coefficient of the highest power of λ in Eq. (2.14) be equal to the sign of the free term. This necessary stability condition has the form

$$(-1)^n \det(A)\,[1 + kcA^{-1}b] > 0 \tag{2.15}$$

By l we denote the number of positive eigenvalues of matrix A. Then, obviously,

$$(-1)^{n-l}\det(A) > 0 \tag{2.16}$$

Using (2.16), instead of inequality (2.15) we obtain

$$(-1)^l\,[1 + kcA^{-1}b] > 0 \tag{2.17}$$

From inequality (2.17) it ensues that

$$cA^{-1}b < -1/k \quad (0 < -1/cA^{-1}b < k) \quad (l = 2p+1) \tag{2.18}$$

$$cA^{-1}b > -1/k \quad (l = 2p)\,(p \text{ is an integer}) \tag{2.19}$$

Let us now return to the investigation of Eq. (2.12). Here we shall be interested in the nonzero roots.

At first let $l = 2p + 1$. Since here $cA^{-1}b \neq 0$, Eq. (2.12) can be represented in the form

$$\varphi(\sigma)/\sigma = -1/cA^{-1}b \tag{2.20}$$

The left hand side of relation (2.20) is a continuous function and, moreover, $\varphi(\sigma)/\sigma \to$ $\to k$ as $\sigma \to 0$. From condition (1.5) it follows that in the presence of inequality (2.18) Eq. (2.20) has at least two roots: a positive one σ_1 and a negative one σ_{-1} (Fig.1).

Now let $l = 2p$. If $cA^{-1}b = 0$, then Eq. (2.12) has only a trivial solution. When $cA^{-1}b \neq 0$, Eq. (2.12) can be written in form (2.20). If $cA^{-1}b < 0$, then, as we see from inequality (2.19), $-1/cA^{-1}b > k$. From condition (1.4) it ensues that Eq. (2.20) has no roots. If $cA^{-1}b > 0$, then $-1/cA^{-1}b < 0$ and, as again follows from condition (1.4), Eq. (2.20) has no roots. Consequently, if $l = 2p$, Eq. (2.11) has the unique solution $x = 0$.

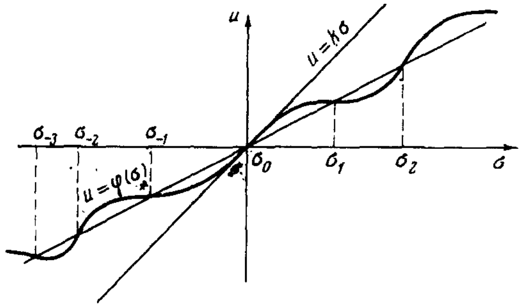

Fig. 1

Thus, taking into account the lemma proved above, we obtain the following two theorems.

Theorem 2.1. More than one stationary state exist in system (1.1), (1.2) if and only if matrix A has no zero eigenvalues and the number of its positive eigenvalues is odd.

Theorem 2.2. If matrix A does not have zero eigenvalues and the number of its positive eigenvalues is odd, then system (1.1), (1.2) has at least three stationary states.

Corollary. If all the eigenvalues of matrix A have nonpositive real parts, then system (1.1), (1.2) has only one equilibrium state $x = 0$. In order to find all the stationary states of the system we need to find all the roots of Eq. (2.12) and to substitute them into the right-hand side of relation (2.13).

As an example consider the following function $\varphi(\sigma)$:

$$u = \varphi(\sigma) = \begin{cases} -M & (k\sigma \leqslant -M) \\ k\sigma & (|k\sigma| \leqslant M) \\ M & (k\sigma \geqslant M) \end{cases} \qquad (M = \text{const} > 0) \tag{2.21}$$

Function (2.21), describing a linear feedback with saturation, obviously satisfies constraints (1.3)–(1.5). When the hypotheses of Theorem 2.2 are fulfilled, Eq. (2.12) has only three solutions,

$$\sigma_{-1} = cA^{-1}bM, \quad \sigma_0 = 0, \quad \sigma_1 = -cA^{-1}bM$$

The three states

$$x_{-1} = A^{-1}bM, \quad x_0 = 0, \quad x_1 = -A^{-1}bM \tag{2.22}$$

will be equilibrium states of system (1.1), (2.21).

We note that the question of the stationary states of controlled systems was analyzed in a somewhat different formulation in [6].

3. The case $k = \infty$. Let us consider the case when the function $\varphi(\sigma)$ is discontinuous at $\sigma = 0$. Let

$$\lim_{\sigma \to -0} \varphi(\sigma) = \varphi(-0) < 0, \qquad \lim_{\sigma \to +0} \varphi(\sigma) = \varphi(+0) > 0$$

We shall assume that the function $\varphi(\sigma)$ is continuous when $\sigma \neq 0$. At $\sigma = 0$ we *define the function $\varphi(\sigma)$ in correspondence with the theory of differential equations* with discontinuous right-hand sides [7]. Let $cb \neq 0$, then

$$\varphi(0) = \begin{cases} \varphi(-0) & \text{when } x \in P_1 \\ -cAx/cb & \text{when } x \in P_2 \\ \varphi(+0) & \text{when } x \in P_3 \end{cases} \tag{3.1}$$

where

$$P_1 = \{x\colon cx = 0, \quad -cAx \,/\, cb < \varphi(-0)\}$$
$$P_2 = \{x\colon cx = 0, \quad \varphi(-0) \leqslant -cAx \,/\, cb \leqslant \varphi(+0)\}$$
$$P_3 = \{x\colon cx = 0, \quad \varphi(+0) < -cAx \,/\, cb\}$$

The definition (3.1) of the value $\varphi(0)$ is such that the derivative $\sigma^{\cdot}$ relative to system (1.1), (1.2) equals zero at points $x \in P_2$. In region P_2 system (1.1), (1.2) has the form

$$\frac{dx}{dt} = \left(A - \frac{bcA}{cb}\right)x \tag{3.2}$$

At the points $x \in P_1$ or $x \in P_3$ the derivative $\sigma^{\cdot} \neq 0$, i.e. the phase trajectories "pierce" the segments P_1 and P_3 of the hyperplane $cx = 0$.

As before we shall take it that conditions (1.4) (the left part) and (1.5) are fulfilled.

As we see from (3.1), the state $x = 0$ is a stationary state of system (1.1), (1.2). We shall assume that this equilibrium state is asymptotically stable. The necessary and sufficient condition for the asymptotic stability of the state $x = 0$ contains two requirements [8].

The first requirement is that the set P_2 should be an "attracting" set. This requirement is described in the form of the inequality

$$cb < 0 \tag{3.3}$$

The second requirement is that a motion starting from points $x \in P_2$ sufficiently close to the origin tends asymptotically to the origin, remaining on set P_2. Equation (3.2), which describes the motion of the system on set P_2 (the so-called sliding motion), obviously has one zero eigenvalue $\lambda_1 = 0$ corresponding to the integral $cx = \text{const}$. The second requirement reduces to this, that all the remaining eigenvalues of system (3.2) have negative real parts,

$$\operatorname{Re}\lambda_i < 0 \qquad (i = 2, 3, \ldots, n) \tag{3.4}$$

The values λ_i $(i = 1, 2, \ldots, n)$ are the roots of the characteristic equation

$$\det\,(A - bcA \,/\, cb - \lambda E) = 0 \tag{3.5}$$

We now proceed directly to the investigation of the question regarding the stationary states. There is only one equilibrium state $x = 0$ on the plane $cx = 0$. Indeed, there are no stationary states on the sets P_1 and P_3 because on them $\sigma^{\cdot} \neq 0$ (by virtue of inequality (3.3) $\sigma^{\cdot} < 0$ when $x \in P_1$ and $\sigma^{\cdot} > 0$ when $x \in P_3$). On the set P_2 there cannot be stationary states besides the state $x = 0$ as a consequence of condition (3.4).

The investigation of the question on equilibrium states lying ontside the plane $cx = 0$ reduces to the examination of Eq. (2.1). Under condition (2.2) the investigation of Eq. (2.1), analogous to the one carried out in Sect. 2, leads to the lemma formulated in that section. Thus this lemma holds also in the case $k = \infty$.

The course of the arguments as presented in the preceding section for condition (2.2) is preserved for condition (2.10). We should set $k = \infty$ in inequalities (2.18) and (2.19). However, let us derive these inequalities for the case $k = \infty$ by means of rigorous arguments. As follows from [3], the characteristic equation (3.5) can be written in the form

$$\det\,(A - \lambda E)\,[1 - (cA/cb)(A - \lambda E)^{-1}\,b] = 0 \tag{3.6}$$

In this equation the free term equals zero. The expansion of the expression occurring within the brackets in powers of λ starts with the term $-\lambda\,(cA^{-1}b/cb)$. For the fulfillment of inequalities (3.4) it is necessary that the sign of the coefficient of the highest power

λ^n in Eq. (3.6) be equal to the sign of the coefficient of λ. This necessary condition has the form

$$(-1)^n \det(A)\,(-cA^{-1}b/cb) > 0 \qquad (3.7)$$

Using inequalities (2.16) and (3.3), instead of (3.7) we obtain the expression

$$(-1)^l cA^{-1}b > 0$$

Hence follow the inequalities which are obtained from (2.18), (2.19) with $k = \infty$

$$cA^{-1}b < 0 \quad (l = 2p+1), \qquad cA^{-1}b > 0 \quad (l = 2p)$$

Thus, Theorems 2.1 and 2.2 hold in the case $k = \infty$.

A typical example for the present section is a system with relay feedback. In such a system the function $\varphi(\sigma)$ has the form

$$\varphi(\sigma) = -M \quad (\sigma < 0), \qquad \varphi(\sigma) = M \quad (\sigma > 0)$$

With due regard to this formula and to formula (3.1) the expression for the control can be written in the form

$$u = \begin{cases} -M & (cx < 0 \ \text{ or } \ cx = 0 \ \text{ and } \ -cAx/cb < -M) \\ -cAx/cb & (cx = 0 \ \text{ and } \ |cAx/cb| \leqslant M) \\ M & (cx > 0 \ \text{ or } \ cx = 0 \ \text{ and } \ -cAx/cb > M) \end{cases} \qquad (3.8)$$

The function $\varphi(\sigma)$ satisfies all the conditions listed in this section. When the hypotheses of Theorem 2.2 are fulfilled, system (1.1), (3.8) has the three stationary states (2.22).

Let us present one more example.

We consider the motion of an aircraft at a constant altitude with a velocity of constant magnitude. We assume that the aircraft has been roll stabilized, then the aircraft's equations of motion in the horizontal plane can be written, under certain assumptions, in the form [9, 10]

$$dx_1/dt = x_2, \quad dx_2/dt = a_{22}x_2 + a_{23}x_3 + b_2u, \quad dx_3/dt = a_{32}x_2 + a_{33}x_3 + b_3u \qquad (3.9)$$

Here x_1 and x_3 are the course and the sideslip angles, x_2 is the angular course velocity, u is the angle of deflection of the aerodynamic surfaces satisfying the condition $|u| \leqslant u_0$, where u_0 is the maximal possible deflection angle, the constant coefficients a_{22}, a_{23}, a_{32}, a_{33}, b_2, b_3 are determined by the aerodynamic and weight design of the aircraft and by the velocity of the motion of the center of gravity.

Let us assume that the desired mode of motion of the aircraft is motion with a constant course angle $x_1 = 0$. Then, obviously, the desired state of system (3.9) will be the state $x = 0$. Suppose that the feedback (1.2) which stabilizes this state satisfies conditions (1.3) (or (3.1)), (1.4), (1.7); as a consequence of the constraint $|u| \leqslant u_0$, it also satisfies condition (1.5). The matrix A in system (3.9) has at least one zero eigenvalue (the aircraft without feedback is indifferent to the course angle). Therefore, system (3.9), (1.2) has only the one stationary state $x = 0$.

4. Stability of the stationary states. In this section we shall assume that the hypotheses of Theorem 2.2 are fulfilled, i.e. we shall take it that $\det(A) \neq 0$ and $l = 2p + 1$.

Let $x_s \neq 0$ be a stationary state of system (1.1), (1.2), corresponding to the roots $\sigma_s \neq 0$ of Eq. (2.12). Let us consider the question of the stability of this stationary state.

By means of the relation

$$x = x_s + y \tag{4.1}$$

we introduce the vector y of new variables. We shall take it that the function $\varphi(\sigma)$ is analytic in some neighborhood of the value σ_s,

$$\varphi(\sigma) = \varphi(\sigma_s) + \varphi'(\sigma_s)(\sigma - \sigma_s) + (1/m!)\varphi^{(m)}(\sigma_s)(\sigma - \sigma_s)^m + O[(\sigma - \sigma_s)^{m+1}] \tag{4.2}$$

Here m is the order of the first, after $\varphi'(\sigma_s)$, nonzero derivative of the function $\varphi(\sigma)$ at the point σ_s and $O[(\sigma - \sigma_s)^{m+1}]$ is a function whose expansion starts with terms of order not less than $m + 1$. Then system (1.1), (1.2) can be written in the form

$$\frac{dy}{dt} = Ay + b\varphi'(\sigma_s)cy + b\frac{1}{m!}\varphi^{(m)}(\sigma_s)(cy)^m + bO[(cy)^{m+1}] \tag{4.3}$$

The question of the stability of the stationary state x_s of system (1.1), (1.2) is reduced to the question of the stability of the equilibrium state $y = 0$ of system (4.3). The first approximation equation for system (4.3) has the form

$$dz / dt = (A + \varphi'(\sigma_s)bc)z \tag{4.4}$$

Let us first consider the situation when the inequality

$$(-1 / cA^{-1}b)\sigma < \varphi(\sigma) \tag{4.5}$$

holds for some $\varepsilon > 0$ and for all values of σ satisfying the condition $\sigma_s - \varepsilon < \sigma < \sigma_s$. A similar situation holds in Fig. 1 at the points σ_{-3}, σ_{-1}, σ_1. Obviously, the inequality

$$-1 / cA^{-1}b \geqslant \varphi'(\sigma_s) \tag{4.6}$$

holds under condition (4.5).

Let us show that the following theorem holds.

Theorem 4.1. If $\varphi'(\sigma_s) < -1 / cA^{-1}b$, then the stationary state x_s of system (1.1), (1.2) is unstable.

System (4.4) differs from system (1.6) only in that in system (4.4), $\varphi'(\sigma_s)$ occurs in the place of the quantity k. Therefore, as follows from inequality (2.18), the inequality $\varphi'(\sigma_s) > -1/cA^{-1}b$ is a necessary condition for the asymptotic stability of system (4.4). As we see from (4.6), this necessary condition is not fulfilled in the situation of (4.5). When strict inequality holds in relation (4.6) (the curve $u = \varphi(\sigma)$ intersects at the point $\sigma = \sigma_s$ the straight line $u = (-1/cA^{-1}b)\sigma$ passing from the upper halfplane to the lower), system (4.4), and, hence, also system (4.3) [11], is unstable.

Suppose that equality holds in relation (4.6) (the curve $u = \varphi(\sigma)$ is tangent to the straight line $u = (-1 / cA^{-1}b)\sigma$) at the point $\sigma = \sigma_s$). Then, as follows from expressions (4.2) and (4.5),

$$\begin{aligned} &\varphi^{(m)}(\sigma_s) > 0, \quad \text{if} \quad m = 2q \\ &\varphi^{(m)}(\sigma_s) < 0, \quad \text{if} \quad m = 2q + 1 \quad (q \text{ is an integer}) \end{aligned} \tag{4.7}$$

When equality holds in relation (4.6), systems (4.3) and (4.4) acquire the form

$$\frac{dy}{dt} = \left(A - \frac{bc}{cA^{-1}b}\right)y + b\frac{1}{m!}\varphi^{(m)}(\sigma_s)(cy)^m + bO[(cy)^{m+1}] \tag{4.8}$$

$$\frac{dz}{dt} = \left(A - \frac{bc}{cA^{-1}b}\right)z \tag{4.9}$$

System (4.9) has a zero eigenvalue corresponding to the integral $cA^{-1}z = \text{const}$. If among the remaining ones there is an eigenvalue with positive real part, then system

(4.9), and, hence, also system (4.8), is unstable.

Let us assume that $\lambda_1 = 0$ is a simple eigenvalue of system (4.9) and that all the remaining eigenvalues have negative real parts,

$$Re\,\lambda_i < 0 \qquad (i = 2, 3, \ldots, n) \tag{4.10}$$

The values λ_i $(i = 1, 2, \ldots, n)$ are the roots of the characteristic equation

$$\det\left(A - bc \,/\, cA^{-1}b - \lambda E\right) = 0 \tag{4.11}$$

We prove the following theorem.

Theorem 4.2. If $\varphi'(\sigma_s) = -1 \,/\, cA^{-1}b$, Eq. (4.11) has a simple zero root, and the real parts of the remaining roots are negative, then the stationary state x_s is unstable.

The assumptions made reduce the problem of the stability of the system (4.8) to the investigation of the so-called [11] critical case of one zero root. Let us apply the results obtained by Liapunov ([11], pp. 92-96) for this critical case to the problem being considered here. In accord with the investigative method suggested in [11] we shall accept the integral $v_1 = cA^{-1}y$ of the linear system (4.9) as the new variable for system (4.8). Furthermore, we introduce new variables $v_2, \ldots, v_n$ such that the corresponding transformation $y = Dv$ is nonsingular. Then, system (4.8) acquires the form

$$\frac{dv}{dt} = D^{-1}\left(A - \frac{bc}{cA^{-1}b}\right)Dv + D^{-1}b\,\frac{1}{m!}\,\varphi^{(m)}(\sigma_s)\,(cDv)^m + D^{-1}bO\left[(cDv)^{m+1}\right] \tag{4.12}$$

Moreover, the first equation of this system (it is obtained by multiplying system (4.8) on the left by the row cA^{-1}) has the form

$$\frac{dv_1}{dt} = cA^{-1}b\,\frac{1}{m!}\,\varphi^{(m)}(\sigma_s)\,(cDv)^m + cA^{-1}bO\left[(cDv)^{m+1}\right] \tag{4.13}$$

We equate the right-hand sides of Eqs. (4.12) to zero,

$$\left(A - \frac{bc}{cA^{-1}b}\right)Dv + b\,\frac{1}{m!}\,\varphi^{(m)}(\sigma_s)\,(cDv)^m + bO\left[(cDv)^{m+1}\right] = 0 \tag{4.14}$$

Let us look upon the last $(n - 1)$ scalar relations in system (4.14) as equations in the variables $v_2, \ldots, v_n$, taking v_1 as an independent variable [11]. Let

$$v_i = u_i(v_1) \qquad (i = 2, 3, \ldots, n) \tag{4.15}$$

be the solution of these equations, where $u_i(0) = 0$.

By multiplying relation (4.14) on the left by the row cA^{-2}, we obtain

$$cA^{-1}Dv - \frac{cA^{-2}b}{cA^{-1}b}\,cDv + cA^{-2}b\,\frac{1}{m!}\,\varphi^{(m)}(\sigma_s)\,(cDv)^m + cA^{-2}bO\left[(cDv)^{m+1}\right] = 0 \tag{4.16}$$

The first term in Eq. (4.16) is the variable v_1. The solution of Eq. (4.16) with respect to the variable cDv can be expanded into a series in the variable v_1 in the following manner:

$$cDv = \frac{cA^{-1}b}{cA^{-2}b}\,v_1 + \ldots \tag{4.17}$$

where the dots denote terms of higher order in the variable v_1. Hence it follows that the quantity $cDv(v_1)$, where $v(v_1)$ is a column consisting of the functions $v_1, u_2(v_1), \ldots, u_n(v_1)$, also can be expanded into series (4.17).

We make a change of variables in system (4.12) by means of the relations

$$v_1 = v_1, \quad v_i = w_i + u_i(v_1) \quad (i = 2, 3, \ldots, n)$$

After this change Eq. (4.13) acquires the form

$$dv_1 / dt = gv_1^m + V_1(v_1, w_2, \ldots, w_n) \tag{4.18}$$

where, in correspondence with (4.17),

$$g = \frac{1}{m!} \varphi^{(m)}(\sigma_s) \frac{(cA^{-1}b)^{m+1}}{(cA^{-2}b)^m} \tag{4.19}$$

the function $V_1(v_1, w_2, \ldots, w_n)$ contains terms whose dimesion is not less than m, and the expansion of the function $V_1(v_1, 0, \ldots, 0)$ starts with terms whose degree is not less than $m+1$.

If $m = 2q$, system (4.8) is unstable [11] independently of the sign of the coefficient g.

If $m = 2q + 1$, the solution of the stability problem does depend on the sign of coefficient g. In expression (4.19) the signs of all the quantities excepting the quantity $cA^{-2}b$ are known. Let us find the sign of this one quantity.

Using the determinantal relation [3], the characteristic equation (4.11) is written in the form

$$\det(A - \lambda E)\left[1 - \frac{c(A - \lambda E)^{-1}b}{cA^{-1}b}\right] = 0 \tag{4.20}$$

In this equation the free term equals zero. The expansion of the expression within the brackets in powers of λ starts with the term $-\lambda(cA^{-2}b/cA^{-1}b)$. From inequalities (4.10) it follows that the sign of the coefficient of λ^n in Eq. (4.20) equals the sign of the coefficient of λ. This condition can be written in the form

$$(-1)^n \det(A)(-cA^{-2}b/cA^{-1}b) > 0 \tag{4.21}$$

Using the condition that $l = 2p + 1$ and the inequalities (2.16) and (2.18), instead of relation (4.21) we obtain the desired inequality

$$cA^{-2}b < 0 \tag{4.22}$$

From inequalities (4.7) and (4.22) it follows that if $m = 2q + 1$, then $g > 0$. Consequently, system (4.8) is unstable when m is odd [11].

The critical cases when Eq. (4.11) has multiple zero roots or has a zero root and a pair of pure imaginary roots, while the real parts of the remaining roots are negative, remain uninvestigated.

We now look at another situation. Suppose that there exists $\varepsilon > 0$ such that the inequality

$$(-1 / cA^{-1}b)\sigma > \varphi(\sigma) \tag{4.23}$$

holds for all values of σ satisfying the condition $\sigma_s - \varepsilon < \sigma < \sigma_s$. A similar situation holds on the Fig. 1 at the points σ_{-2}, σ_2. From condition (4.23) we obtain that

$$-1 / cA^{-1}b \leqslant \varphi'(\sigma_s) \tag{4.24}$$

If strict inequality holds in relation (4.24) (the curve $u = \varphi(\sigma)$ intersects at the point $\sigma = \sigma_s$ the straight line $u = (-1 / cA^{-1}b)\sigma$ passing from the lower halfplane to the upper), then this signifies that only the necessary stability condition (2.18) is fulfilled. We cannot make a successful conclusion regarding the stability in this case. For example, the state x_s is obviously asimptotically stable if $\varphi'(\sigma_s) = k$. As follows from [1], the state x_s will be asymptotically stable also for all values of $\varphi'(\sigma_s)$ sufficiently close to the value k.

We now examine the case when equality holds in relation (4.24). Here, as follows from expressions (4.2) and (4.23),

$$\begin{aligned} \varphi^{(m)}(\sigma_s) < 0, &\quad \text{if} \quad m = 2q \\ \varphi^{(m)}(\sigma_s) > 0, &\quad \text{if} \quad m = 2q + 1 \end{aligned} \tag{4.25}$$

In the case being considered systems (4.3) and (4.4) acquire the form (4.8) and (4.9), respectively. We shall assume that inequality (4.10) holds and, here, the stability problem reduces to the investigation of the critical case of one zero root. In this case stability is determined by the quantities m and g in Eq. (4.18). When m is even the state x_s is unstable [11].

Now let $m = 2q + 1$. Here, as follows from inequalities (4.22) and (4.25), the coefficient $g < 0$ and, consequently [11], the stationary state x_s is asymptotically stable.

Thus, the following theorem holds in the situation of (4.23).

Theorem 4.3. If $\varphi'(\sigma_s) = -1 / cA^{-1}b$, Eq. (4.11) has a simple zero root while the real parts of the remaining roots are negative, then the stationary state x_s of system (1.1), (1.2) is unstable for even m and is asymptotically stable for odd m.

Thus, in the situation of (4.23) the state x_s can be both asymptotically stable and unstable depending on the behavior of the function $\varphi(\sigma)$.

For the Eqs. (2.21) and (3.8), presented as examples in the preceding sections, the stationary states $x_{-1} = A^{-1}bM$ and $x_1 = -A^{-1}bM$ are unstable. This follows from Theorem 4.1.

5. Stability region. Let W denote the region of attraction (the stability region) of the origin, i.e. the set of states x from which system (1.1), (1.2) asymptotically goes into the origin. Obviously, there exists a neighborhood S of the origin, belonging wholly to region W.

If system (1.1), (1.2) has only one equilibrium state $x = 0$, this still does not mean that the region W coincides with the whole space X. If, however, the system has more than one stationary state, then we can assert that region W does not coincide with the whole space X.

From Theorem 2.2 it follows that if matrix A is nonsingular and if the number of its positive eigenvalues is odd, then the region W does not coincide with the whole space X, i.e. system (1.1), (1.2) is not stable "in-the-large".

If function (1.2) is bounded, $|\varphi(\sigma)| \leqslant M$, as in examples (2.21), (3.8) and (3.9), then the region W occupies only a part of space X in any case when there are eigenvalues with positive real part in matrix A. Indeed, as follows from [2, 12] for example, in such a case the so-called region Q of controllability of system (1.1) with controls $|u(t)| \leqslant M$ occupies only a part of space X, and $W \subset Q$.

We take an arbitrary point $x_0 \in W$. There exists an instant T such that the trajectory $x(x_0, t)$ of system (1.1), (1.2), starting from the point x_0, turns out to be inside the neighborhood S at this instant, $x(x_0, T) \in S$. From the fact of the continuous dependence of the solution on the initial conditions if follows that the trajectories starting from a sufficiently small neighborhood of the point x_0, turn out to be in the neighborhood S at the instant T. It follows from this that the region W is open.

Let R denote the set of limit (boundary) points of region W. If $x_0 \in R$, then $x(x_0, t) \in R$ for all t. This follows readily from the continuous dependence of the solution on the initial conditions. In other words, the boundary R of the stability region W consists wholly of integral trajectories of system (1.1), (1.2).

It is not difficult to see that only the unstable stationary states of system (1.1), (1.2) may belong to the boundary R.

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SELF-SIMILAR SOLUTIONS OF THE BELLMAN EQUATION FOR OPTIMAL CORRECTION OF RANDOM DISTURBANCES

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A nonlinear second order partial differential equation (Bellman equation) is solved for some characteristic problems of optimal correction of motion in the presence of random distrubances and integral constraints on the control function.

For these problems, classes of self-similar (invariant group) solutions of the Bellman equation are computed. Some exact analytical solutions are obtained.

1. Formulation of problem. Let the motion of the system be described by the following equation:

$$dx / dt = a(t)u + b(t)\xi, \qquad x(t_0) = x_0 \tag{1.1}$$

Here t is time, x is the scalar phase coordinate, u is the control function, ξ is the random disturbance which is represented by white noise of constant intensity, $a(t)$ and $b(t)$ are given functions of time which have the meaning of control efficiency and